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Determinant formula for the six-vertex model

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Abstract. The partition function of a six-vertex model with domain wall boundary conditions is considered on the finite lattice. We show that the partition function satisfies a recursion relation. We solve the recursion relation by a determinant formula. This gives a determinant representation for the partition function. We use the Quantum Inverse Scattering Method (QISM).

1. Introduction

Six-vertex (ice-type) models were first solved explicitly in the thermodynamic limit [15–18]. The model was solved in the sense of finding an explicit expression for the partition function in the large lattice limit. This analysis also relied on periodic boundary conditions [15]. This allowed for the analysis of several thermodynamic quantities. Our analyis of the six-vertex model will dispense with the requirement for a large lattice by analysing a different set of boundary conditions, namely domain wall boundary conditions. The interested reader should refer to the following articles for more information concerning detailed analyses of the six-vertex model in the thermodynamic limit [4, 5, 9, 19, 21, 23].

In [1], the inhomogeneous version of the six-vertex model was introduced and solved in the thermodynamic limit. In this paper, we present a detailed analysis of the partition function of the inhomogeneous six-vertex model with domain wall boundary conditions on the square lattice. QISM is used to formulate the partition function and determine its recursive properties. This recursion relation was first derived for the model on the square lattice in [13]. The recursion relation for the partition function is solved by a determinant formula. In [2], this recursion relation was presented for a more general situation. In [8], Izergin made a short communication of the determinant formula for the partition function of the six-vertex model on the square lattice.

Baxter has recently formulated a partition function for the Z invariant six-vertex model in a finite size box with specific boundary conditions: namely, half the spins point in and the other half point out [2]. This is the natural generalization of domain

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wall boundary conditions. It is our hope that an explicit formulation of the partition function for domain wall boundary conditions presented in this paper will help solve the more general case put forward by Baxter.

Domain wall boundary conditions appear naturally in the study of correlation functions. The determinant representation of this paper allows one to write down the determinant representation for quantum-correlation functions. This can then be used to obtain differential equations which can then be solved to give explicit expressions for quantum-correlation functions [6, 7, 14]. We also hope that this formula can be used in knot theory since statistical physics of exactly solvable models is closely related to knot theory and the Braid group, [22] and references therein. As an example, the well known polynomial of Jones [10] can be considered as a partition function of some statistical system [11, 12].

This paper is organized as follows. In section 2, the six-vertex model is reviewed. In section 3, we translate the six-vertex model into the language of the QISM and define the partition function. In section 4, we prove the recursion relations for the partition function and we also prove that they define the partition function in a unique way. The determinant formula for the partition function is presented in section 5. In section 6, we present, in addition, the partition function for the six-vertex model for the special case with rational statistical weights. Two-dimensional models of classical-statistical physics (which we are considering) have one-dimensional quantum counterparts. These are the Heisenberg magnets. The six-vertex model generates the XXZ Heisenberg magnet. A special case with rational statistical weights (which was just mentioned) generates the XXX Heisenberg magnet. Details concerning these Heisenberg magnets are contained in the appendix. Finally, in section 7, we analyse the homogeneous limit of the partition function from section 5. In section 8, the results are summarized. In the appendix, we review the QISM in the context of the XXX and XXZ Heisenberg models.

2. Six-vertex model

In this section, we will briefly review the six-vertex model as a model of interacting spins in two dimensions. The inhomogeneous version was formulated by Baxter [1]. In this paper, we solve the model on a finite ($N \times N$) lattice with domain wall boundary conditions. To accomplish this, we will make a translation of the model from the language of statistical physics [3] to the language of QISM in the next section.

The partition function is defined by the following:

$$Z = \sum \exp\left[-\frac{E}{kT}\right] \tag{2.1}$$

where E is the energy of the system and the summation is over all possible configuations of the system. The model has six possible vertex configurations represented by arrows going into or out of a vertex, see figure 1. The domain wall boundary conditions then correspond to the arrows pointing inward on the top and bottom of the lattice and the arrows pointing outward on the left and right of the lattice, figure 2(a). To compute the partition function, the vertices must be assigned statistical weights. Following [3], we will make the following restriction on these weights

$$a \equiv w_1 = w_2 \qquad b \equiv w_3 = w_4 \qquad c \equiv w_5 = w_6 \tag{2.2}$$

which makes the model invariant under a simultaneous reversal of all arrows. Hence, the partition function can be rewritten as

$$Z = \sum a^{n_1+n_2} b^{n_3+n_4} c^{n_5+n_6} \tag{2.3}$$

where the summation is over all possible vertex configurations with n_i being the number of vertices of type i.

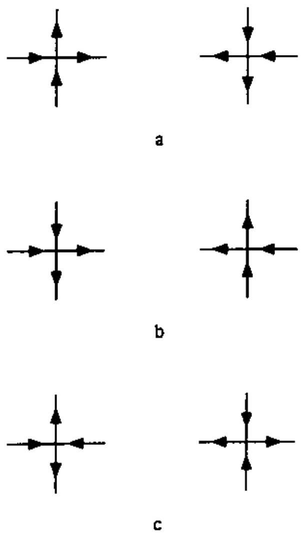

Figure 1. Vertex configurations and their associated weights.

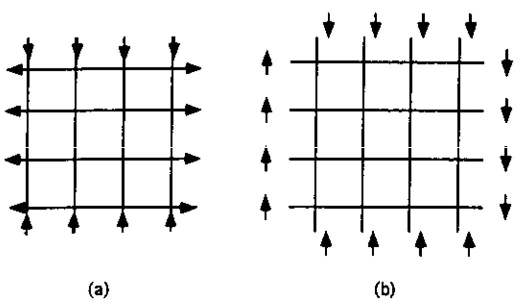

Figure 2. Domain wall boundary conditions: (*a*) statistical model and (*b*) QISM formulation.

In the inhomogeneous model, the statistical weights become site dependent. To make this explicit, two sets of spectral parameters will be used, $\{l_\alpha\}$ and $\{\nu_k\}$, see figure 3. These sets of spectral parameters are in one to one correspondence to the set of lines. l_1 corresponds to the first horizontal line, l_2 corresponds to the second horizontal line, l_β corresponds to the βth horizontal line, etc. Similarly ν_1 corresponds to the first vertical line, ν_2 corresponds to the second vertical line, ν_l corresponds to the lth vertical line, etc. The horizontal rows will be enumerated by Greek indices, ($\alpha = 1, \ldots, N$) with parameters $\{l_\alpha\}$. The vertical columns will be enumerated by Latin indices, ($k = 1, \ldots, N$) with parameters $\{\nu_k\}$. Each statistical weight is associated with a site of the lattice and each site is associated with the intersection of two lines. Consequently, the statistical weight will depend on the two spectral parameters (l_α, ν_k) of these lines. It should be emphasized that the spectral parameters do not obey any constraint, i.e. no Bethe equations.

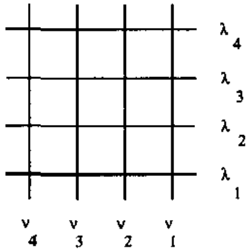

Figure 3. Relation of spectral parameters to lattice.

A convenient parameterization of the weights at each vertex is

$$\begin{aligned} a(l_\alpha, \nu_k) &= \sinh(l_\alpha - \nu_k + \mu) \\ b(l_\alpha, \nu_k) &= \sinh(l_\alpha - \nu_k - \mu) \\ c(l_\alpha, \nu_k) &= -\sinh(2\mu). \end{aligned} \tag{2.8}$$

The partition function, Z, is now a function of $2N$ variables $\{l_\alpha\}$ and $\{\nu_k\}$.

3. Construction of the partition function with QISM

The QISM formulation of the inhomogeneous six-vertex model involves the use of an L-operator whose matrix elements are the statistical weights (2.4). This L-operator is site dependent and has the following form

$$\mathsf{L}_{\alpha k}(l_\alpha, \nu_k)^{r_k s_k}_{r_\alpha s_\alpha} = \begin{pmatrix} b(l_\alpha, \nu_k) & 0 & 0 & 0 \\ 0 & a(l_\alpha, \nu_k) & c(l_\alpha, \nu_k) & 0 \\ 0 & c(l_\alpha, \nu_k) & a(l_\alpha, \nu_k) & 0 \\ 0 & 0 & 0 & b(l_\alpha, \nu_k) \end{pmatrix} \tag{3.1}$$

where r and s are the spin variables. Here α is the number of the horizontal line and k is the number of the vertical line. r_α and s_α correspond to spins on the horizontal edge α, r_k and s_k correspond to spins on the vertical edge k, figure 5. In this language, we no longer consider arrows as going into or out of a vertex. We must now consider spins on an edge as pointing up or down. More explicitly, for horizontal lines, a left arrow corresponds to an up spin while a right arrow corresponds to a down spin. For vertical lines, an up arrow corresponds to an up spin and a down arrow corresponds to a down spin, see figure 4. Hence we can think of the L-operator (3.1) as the statistical weight when the spins on the surrounding edges are specified, see figures 5 and 6. The L-operator (3.1) is the set of statistical weights associated with the site of the lattice at the intersection of the αth and kth lines.

The **L**-operator (3.1) obeys the intertwining relation in the vertical spin and horizontal spin spaces separately

$$\widetilde{\mathbf{R}}_{\alpha\beta}(l_\alpha, l_\beta)\mathbf{L}_{\alpha k}(l_\alpha - \nu_k)\mathbf{L}_{\beta k}(l_\beta - n_k) = \mathbf{L}_{\beta k}(l_\beta - \nu_k)\mathbf{L}_{\alpha k}(l_\alpha - \nu_k)\widetilde{\mathbf{R}}_{\alpha\beta}(l_\alpha, l_\beta) \quad (3.2)$$

and

$$\widetilde{\mathbf{R}}_{kl}(\nu_l, \nu_k)\mathbf{L}_{\alpha k}(l_\alpha - \nu_k)\mathbf{L}_{\alpha l}(l_\alpha - \nu_l) = \mathbf{L}_{\alpha l}(l_\alpha - \nu_l)\mathbf{L}_{\alpha k}(l_\alpha - \nu_k)\widetilde{\mathbf{R}}_{kl}(\nu_l, \nu_k) \quad (3.3)$$

where the matrix $\widetilde{\mathbf{R}}$ is defined in the appendix, see (A4), and is related to the **R**-matrix (A12) via $\widetilde{\mathbf{R}} = \mathbf{PR}$ where **P** is the permutation matrix (A5). The **R**-matrix satisfies the Yang–Baxter relation and acts non-trivially in the tensor product space of two linear spaces. Subindices of the **R**-matrix are the numbers of these spaces. The monodromy matrix, product of **L**-operators through a line of the lattice, obeys a similar intertwining relation with **R** (see (3.9) and (A10)). It should be noted that there is a monodromy matrix associated with each vertical column and each horizontal row.

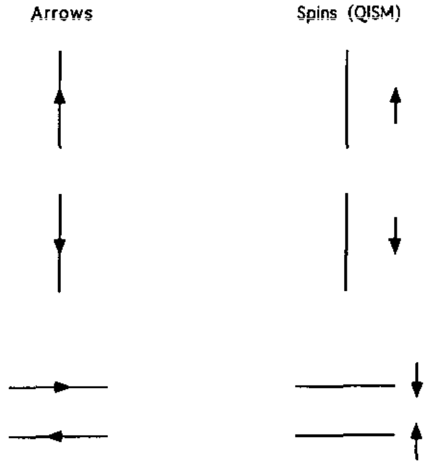

Figure 4. Translation between arrows in the statistical six-vertex formulation and spins in the QISM formulation.

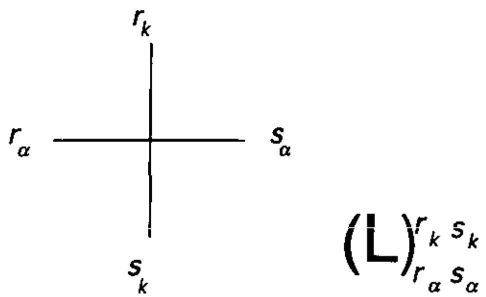

Figure 5. L-operator associated with each vertex.

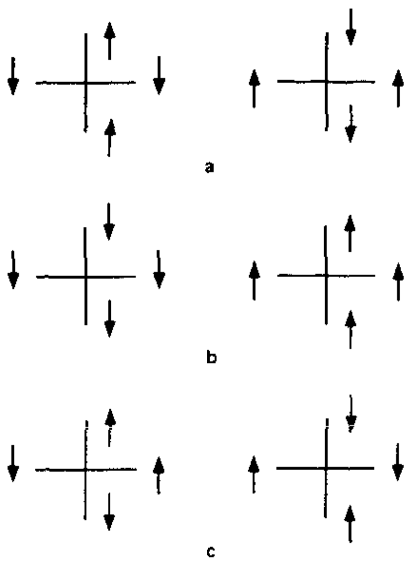

Figure 6. Vertex configurations in the QISM formulation.

In what follows, we will make the following change of variables: $\mathrm{i}\eta = \mu$. The L-operator can then be written explicitly in a more convenient form

$$\mathbf{L}_{\alpha k}(l_\alpha - \nu_k) = \cos\eta \sinh(l_\alpha - \nu_k) - \mathrm{i}\sin\eta\cosh(l_\alpha - \nu_k)\sigma_z^\alpha\sigma_z^k \\ - \mathrm{i}\sin 2\eta(\sigma_-^\alpha\sigma_+^k + \sigma_+^\alpha\sigma_-^k) \tag{3.4}$$

which is again the statistical weight at the vertex of the kth column and the αth row. Likewise, l_α is the spectral parameter associated with the horizontal line α and ν_k is the spectral parameter associated with the vertical line k. The upper index of the σ matrices indicates the space in which the L-operator acts non-trivially.

This L-operator (3.4) has two eigenvectors $(\uparrow_\alpha\uparrow_k)$ and $(\downarrow_\alpha\downarrow_k)$ where $\uparrow_\alpha$ corresponds to an up spin in the αth (horizontal) space and $\uparrow_k$ corresponds to an up spin in the kth (vertical) space. The eigenvalues are

$$\begin{aligned}\mathbf{L}_{\alpha k}(l_\alpha - \nu_k)(\uparrow_\alpha\uparrow_k) &= \sinh(l_\alpha - \nu_k - \mathrm{i}\eta)(\uparrow_\alpha\uparrow_k)\\ \mathbf{L}_{\alpha k}(l_\alpha - \nu_k)(\downarrow_\alpha\downarrow_k) &= \sinh(l_\alpha - \nu_k - \mathrm{i}\eta)(\downarrow_\alpha\downarrow_k).\end{aligned} \tag{3.5}$$

In our treatment of this spin system, we are considering domain wall boundary conditions. In this case, the $N \times N$ lattice remains finite in extent and the spins on the upper and right-hand sides of the lattice point down while the spins on the lower and left-hand sides of the lattice point up, figure 2(*b*). This leads to the following expression for the partition function in terms of the L-operator

$$\begin{aligned}Z_N(\{l_\alpha\},\{\nu_k\}) &= \sum_{\text{spins}} \prod_{\text{vertices}} (\mathbf{L}_{\alpha k}(l_\alpha - \nu_k))^{r_k s_k}_{r_\alpha s_\alpha}\\ &= \left\{\prod_{\beta=1}^{N} \uparrow_\beta \prod_{j=1}^{N} \downarrow_j\right\}\left\{\prod_{\alpha=1}^{N}\prod_{k=1}^{N} \mathbf{L}_{\alpha k}(l_\alpha - \nu_k)\right\}\left\{\prod_{\beta=1}^{N} \downarrow_\beta \prod_{j=1}^{N} \uparrow_j\right\}.\end{aligned} \tag{3.6}$$

Here the double product shall be taken as space ordered

$$\prod_{\alpha=1}^{N}\prod_{k=1}^{N}\mathsf{L}_{\alpha k}=(\mathsf{L}_{NN}\cdots\mathsf{L}_{N2}\mathsf{L}_{N1})\cdots(\mathsf{L}_{2N}\cdots\mathsf{L}_{22}\mathsf{L}_{21})(\mathsf{L}_{1N}\cdots\mathsf{L}_{12}\mathsf{L}_{11}). \tag{3.7}$$

In order to further understand the stucture of the partition function, we should examine the monodromy matrix, product of L-operators, along each horizontal and vertical line. Along the αth horizontal line, the monodromy matrix $\mathsf{T}_{\alpha}(l_{\alpha})$ is given by

$$\mathsf{T}_{\alpha}(l_{\alpha})=\prod_{k=1}^{N}\mathsf{L}_{\alpha k}(l_{\alpha}-\nu_{k})=\begin{pmatrix}A(l_{\alpha}) & B(l_{\alpha})\\ C(l_{\alpha}) & D(l_{\alpha})\end{pmatrix}. \tag{3.8}$$

It is important to note that this matrix obeys the intertwining relation

$$\widetilde{\mathsf{R}}_{\alpha\beta}(l_{\alpha},l_{\beta})\mathsf{T}_{\alpha}(l_{\alpha})\mathsf{T}_{\beta}(l_{\beta})=\mathsf{T}_{\beta}(l_{\beta})\mathsf{T}_{\alpha}(l_{\alpha})\widetilde{\mathsf{R}}_{\alpha\beta}(l_{\alpha},l_{\beta}). \tag{3.9}$$

The monodromy matrix along the jth vertical line $\tau_{j}(\nu_{j})$ is given by

$$\tau_{j}(\nu_{j})=\prod_{\alpha=1}^{N}\mathsf{L}_{\alpha j}(l_{\alpha}-\nu_{j})=\begin{pmatrix}A(\nu_{j}) & B(\nu_{j})\\ C(\nu_{j}) & D(\nu_{j})\end{pmatrix}. \tag{3.10}$$

which also satisfies a relation similar to (3.9)

$$\widetilde{\mathsf{R}}_{kl}(\nu_{l},\nu_{k})\tau_{k}(\nu_{k})\tau_{l}(\nu_{l})=\tau_{l}(\nu_{l})\tau_{k}(\nu_{k})\widetilde{\mathsf{R}}_{kl}(\nu_{l},\nu_{k}). \tag{3.11}$$

The matrix elements A, B, C and D are operators, see appendix. Hence the partition function (3.6) can be rewritten as

$$\begin{aligned}Z_{N}&=\left\{\prod_{j=1}^{N}\downarrow_{j}\right\}\left\{\prod_{\alpha=1}^{N}\uparrow_{\alpha}\mathsf{T}_{\alpha}(l_{\alpha})\prod_{\alpha=1}^{N}\downarrow_{\alpha}\right\}\left\{\prod_{j=1}^{N}\uparrow_{j}\right\}\\ &=\left\{\prod_{j=1}^{N}\downarrow_{j}\right\}\left\{\prod_{\alpha=1}^{N}B(l_{\alpha})\right\}\left\{\prod_{j=1}^{N}\uparrow_{j}\right\}\end{aligned} \tag{3.12}$$

or

$$\begin{aligned}Z_{N}&=\left\{\prod_{\beta=1}^{N}\uparrow_{\beta}\right\}\left\{\prod_{j=1}^{N}\downarrow_{j}\tau_{j}(\nu_{j})\prod_{j=1}^{N}\uparrow_{j}\right\}\left\{\prod_{\beta=1}^{N}\downarrow_{\beta}\right\}\\ &=\left\{\prod_{\beta=1}^{N}\uparrow_{\beta}\right\}\left\{\prod_{j=1}^{N}C(\nu_{j})\right\}\left\{\prod_{\beta=1}^{N}\downarrow_{\beta}\right\}.\end{aligned} \tag{3.13}$$

Thus we have obtained an expression for the partition function of the six-vertex model in terms of QISM. These formulae will be used in the next section to determine a set of lemmas which will uniquely determine the explicit expression for Z_{N}. More information on the operator elements of the monodromy matrix can be found in the appendix.

4. Recursion relation for Z_N

In this section, we present 4 lemmas that the partition function (3.6) satisfies. These were first established by V E Korepin [13]. It should be emphasized that these properties uniquely determine the partition function.

Lemma. (a) $Z_1 = -\mathrm{i}\sin 2\eta$.

Proof. This follows directly from (3.6) by setting $N = 1$.

Lemma. (b) Z_N is a symmetric function of l_α ($\alpha = 1, \ldots, N$) and ν_j ($j = 1, \ldots, N$) separately.

Proof. The symmetry of Z_N can be seen from (3.12) and (3.13) since interchanging $l_\beta \leftrightarrow l_\gamma$ corresponds to the interchange of $B(l_\beta) \leftrightarrow B(l_\gamma)$. However, $[B(l_\beta), B(l_\gamma)] = 0$, following from (3.9), leaving the partition function unchanged. The same holds for $\nu_j \leftrightarrow \nu_k$ since $[C(\nu_j), C(\nu_k)] = 0$ which follows from (3.11).

Lemma. (c) Z_N is a function in each variable l_α and ν_j of the form

$$Z_N = \eta^{[-(N-1)l_\alpha]} P_{N-1}(\exp(2l_\alpha)) = \eta^{[(N-1)\nu_j]} Q_{N-1}(\exp(-2\nu_j)) \tag{4.1}$$

where P_{N-1} and Q_{N-1} are polynomials of degree $N-1$ in their respective arguments.

Proof. This lemma can be shown by computing

$$\frac{\partial^N}{\partial(\mathrm{e}^{2l_\alpha})^N} \eta^{[(N-1)l_\alpha]} \mathrm{T}_\alpha(l_\alpha)$$

which leads to

$$\frac{\partial^N}{\partial(\mathrm{e}^{2l_\alpha})^N} \eta^{[(N-1)l_\alpha]} B_\alpha(l_\alpha) = 0.$$

Thus (c) is fulfilled since the dependence of Z_N on l_α enters only by means of $B(l_\alpha)$. A similar result holds for each ν_j.

Lemma. (d) Z_N obeys the following recursion relation

$$Z_N(\{l_\alpha\}, \{\nu_j\})\big|_{l_\beta - \nu_l = -\mathrm{i}\eta} = -\mathrm{i}\sin 2\eta \left[\prod_{k \neq l}^{N} \sinh(l_\beta - \nu_k - \mathrm{i}\eta)\right] \times \left[\prod_{\alpha \neq \beta}^{N} \sinh(l_\alpha - \nu_l - \mathrm{i}\eta)\right] Z_{N-1}(\{l_{\alpha\neq\beta}\}, \{\nu_{j\neq l}\}). \tag{4.2}$$

Proof. To verify this, it is sufficient to set $\beta = l = 1$ using (b). Noting that L_{11} is furthest to the right in (3.6), we can determine

$$\mathsf{L}_{11}(l_1 - \nu_1 = -\mathrm{i}\eta)\left\{\prod_{\alpha=1}^{N} \downarrow_\alpha \prod_{j=1}^{N} \uparrow_j\right\} = -\mathrm{i}\sin 2\eta \left\{\uparrow_1 \prod_{\alpha=2}^{N} \downarrow_\alpha\right\}\left\{\downarrow_1 \prod_{j=2}^{N} \uparrow_j\right\}. \tag{4.3}$$

The vector remaining on the right-hand side of (4.3) is an eigenvector of each $\mathsf{L}_{1k}(l_1 - \nu_k)$ $(k = 2, \ldots, N)$ with eigenvalue $\sinh(l_1 - \nu_k - \mathrm{i}\eta)$ and an eigenvector of each $\mathsf{L}_{\alpha 1}(l_\alpha - \nu_1)$ $(\alpha = 2, \ldots, N)$ with eigenvalue $\sinh(l_\alpha - \nu_1 - \mathrm{i}\eta)$, see (3.5). Thus we are led to the following result

$$Z_N(\{l_\alpha\}, \{\nu_j\})\big|_{l_1-\nu_1=-\mathrm{i}\eta} = -\mathrm{i}\sin 2\eta \left[\prod_{k=2}^{N} \sinh(l_1 - \nu_k - \mathrm{i}\eta)\right] \times \left[\prod_{\alpha=2}^{N} \sinh(l_\alpha - \nu_1 - \mathrm{i}\eta)\right] Z_{N-1}(\{l_{\alpha\neq 1}\}, \{\nu_{j\neq 1}\}). \tag{4.4}$$

The product of $\sinh(l_1 - \nu_k - \mathrm{i}\eta)$ appears because we applied the product of all L_{1k} from (3.6). This proves (d).

It is now important to state that the 4 lemmas given above determine Z_N uniquely. This can be shown by induction starting with the result for Z_1. Since Z_N as a function of $\exp(2l_\alpha)$ is a polynomial of degree $(N-1)$, there must be N equations to fix the N coefficients. The recursion relation (4.2) gives the value of this polynomial at N points

$$l_\beta = \nu_1 - \mathrm{i}\eta \qquad l_\beta = \nu_2 - \mathrm{i}\eta \qquad \cdots l_\beta = \nu_N - \mathrm{i}\eta$$

providing the necessary equations. In the next section, these properties will be used to verify that the partition function Z_N (3.6) can be written as a determinant.

5. Determinant formula for Z_N

The following formula for the partition function solves the recursion relation presented in the previous section

$$Z_N(\{l_\alpha\}, \{\nu_j\}) = (-1)^N \prod_{\alpha=1}^{N}\prod_{k=1}^{N} \sinh(l_\alpha - \nu_k - \mathrm{i}\eta)\sinh(l_\alpha - \nu_k + \mathrm{i}\eta) \times \left[\prod_{1\leqslant\alpha<\beta\leqslant N} \sinh(l_\alpha - l_\beta) \prod_{1\leqslant k<l\leqslant N} \sinh(\nu_l - \nu_k)\right]^{-1} \det \mathcal{M} \tag{5.1}$$

where

$$\mathcal{M}_{\alpha k} = \frac{\mathrm{i}\sin 2\eta}{\sinh(l_\alpha - \nu_k - \mathrm{i}\eta)\sinh(l_\alpha - \nu_k + \mathrm{i}\eta)}. \tag{5.2}$$

Proving (5.1) to be the correct expression of the partition function is equivalent to showing that Z_N satisfies the four lemmas from section 4.

(a) This statement can be shown easily by setting $N = 1$ in (5.1). The double product in the numerator and the determinant each contribute only a single term while the denominator contributes no terms. Thus, $Z_1 = -\mathrm{i}\sin 2\eta$.

(b) The symmetry of Z_N in $\{l_\alpha\}$ is easily seen by exchanging $l_\alpha \leftrightarrow l_\beta$ in (5.1). This leads to a factor of -1 from the term

$$\prod_{1\leqslant\alpha<\beta\leqslant N} \sinh(l_\alpha - l_\beta)$$

in the denominator. However, this exchange of ls is equivalent to interchanging two rows in $\mathcal{M}$ which also gives rise to a factor of -1 in $\det\mathcal{M}$. Thus, Z_N is unaltered by exchanging two ls. A similar argument holds for exchanging two νs. Hence, Z_N is a symmetric function in both $\{l_\alpha\}$ and $\{\nu_j\}$ separately.

(c) Proving this for (5.1) is equivalent to showing that the quantity

$$Z'_N \equiv \eta^{[(N-1)l_\alpha]} Z_N \tag{5.3}$$

is a polynomial of degree $(N-1)$ in $\exp(2l_\alpha)$. It is sufficient to prove this for $\alpha = 1$ due to (b). This proof requires two parts. First show that Z'_N is a polynomial rather than a rational function and then show that it is of degree $(N-1)$ in $\exp(2l_1)$.

To prove (5.3) is a polynomial rather than a rational function, we should demonstrate that Z'_N has zero residue at its poles. An analysis of (5.1) and (5.3) shows that there are two sources of poles. The first source is in $\mathcal{M}$ where the poles are zeros of

$$\sinh(l_\alpha - \nu_k + \mathrm{i}\eta)\sinh(l_\alpha - \nu_k - \mathrm{i}\eta).$$

However the poles of $\mathcal{M}$ are also zeros of the numerator of the first factor in (5.1) which implies they have zero residue. The second source of poles is in the denominator of (5.1) when a term in

$$\prod_{1\leqslant\alpha<\beta\leqslant N} \sinh(l_\alpha - l_\beta)$$

is zero. This can be rewritten as the statement that two spectral parameters are related by

$$l_\alpha - l_\beta = \mathrm{i}n\pi \quad (n = 0, 1). \tag{5.4}$$

In this case, the matrix $\mathcal{M}$ will have two equal rows and $\det\mathcal{M} = 0$. Thus, the zeros in the denominator are cancelled by zeros in $\det\mathcal{M}$. Hence, (5.3) is a polynomial rather than a rational function in $\exp(2l)$.

To find the degree of (5.3), it is sufficient to use $\alpha = 1$ as stated above. The double product in the numerator is of order $\exp(2Nl_1)$ while the denominator is of order $\exp((N-1)l_1)$. In $\mathcal{M}$, $\exp(l_1)$ only occurs in the first row which gives a total contribution of $\exp(-2l_1)$. The additional factor of $\exp((N-1)l_1)$ in (5.3) leads to a final result of $\exp(2(N-1)l_1)$ which shows that Z'_N is of degree $(N-1)$ in $\exp(2l_1)$. A similar argument holds for ν_1. Hence, Z_N is an expression of the form (4.1) when all other spectral parameters are taken to be fixed.

(d) To show that (5.1) satisfies the recursion relation (4.2), we will use the symmetry property of Z_N, set $\beta = l = 1$ and prove the following

$$Z_N(\{l_\alpha\},\{\nu_j\})\big|_{l_1-\nu_1=-\mathrm{i}\eta} = -\mathrm{i}\sin 2\eta \left[\prod_{k=2}^{N}\sinh(l_1-\nu_k-\mathrm{i}\eta)\right] \times \left[\prod_{\alpha=2}^{N}\sinh(l_\alpha-\nu_1-\mathrm{i}\eta)\right] Z_{N-1}(\{l_{\alpha\neq 1}\},\{\nu_{j\neq 1}\}). \tag{5.5}$$

The first task at hand is to separate from (5.1) the factors containing l_1 and ν_1. This results in a factorized form for Z_N

$$\begin{aligned} Z_N(\{l_\alpha\},\{\nu_j\}) &= (-1)^N \sinh(l_1-\nu_1-\mathrm{i}\eta)\sinh(l_1-\nu_1+\mathrm{i}\eta) \\ &\times \frac{\prod_{\alpha=2}^{N}\sinh(l_\alpha-\nu_1+\mathrm{i}\eta)\sinh(l_\alpha-\nu_1-\mathrm{i}\eta)}{\prod_{\beta=2}^{N}\sinh(l_1-l_\beta)} \\ &\times \frac{\prod_{k=2}^{N}\sinh(l_1-\nu_k+\mathrm{i}\eta)\sinh(l_1-\nu_k-\mathrm{i}\eta)}{\prod_{l=2}^{N}\sinh(\nu_l-\nu_1)} \\ &\times \frac{\prod_{\alpha=2}^{N}\prod_{k=2}^{N}\sinh(l_\alpha-\nu_k+\mathrm{i}\eta)\sinh(l_\alpha-\nu_k-\mathrm{i}\eta)}{\prod_{2\leqslant\alpha<\beta\leqslant N}\sinh(l_\alpha-l_\beta)\prod_{2\leqslant k<l\leqslant N}\sinh(\nu_l-\nu_k)} \det\mathcal{M}. \end{aligned} \tag{5.6}$$

Here the contribution from l_1 and ν_1 has been isolated everywhere except in $\mathcal{M}$. To accomplish this, we need to understand how $\det\mathcal{M}$ behaves in the limit $l_1 \to \nu_1 - \mathrm{i}\eta$. Examining (5.2), it is easy to verify that in this limit $\mathcal{M}_{11}$ has a pole which dominates the determinant leading to the following factorization, accurate to $\mathcal{O}(1)$

$$\det\mathcal{M} = \det\mathcal{M}_{N-1}\,\frac{-1}{\sinh(l_1-\nu_1+\mathrm{i}\eta)}\bigg|_{l_1-\nu_1=-\mathrm{i}\eta} \tag{5.7}$$

where $\mathcal{M}_{N-1}$ is the $N-1 \times N-1$ minor of $\mathcal{M}$ which is independent of l_1 and ν_1. The pole in (5.7) is cancelled by a zero in the numerator of (5.6) which leads to (5.5) upon setting $l_1 = \nu_1 - \mathrm{i}\eta$ to cancel the additional factors in the second line.

We see that formula (5.1) satisfies all lemmas of section 4. Thus, (5.1) is the correct expression for the partition function of the six-vertex model with domain wall boundary conditions. Since the spectral parameters are arbitrary, this determinant formula is the explicit result for Z_N and can be evaluated for any set of $\{l_\alpha,\nu_k\}$. A similar determinant formula was found for the spin–spin-correlation function of the Ising model which describes the interaction of free fermions [20].

6. Rational six-vertex model

The rational six-vertex model corresponds to the special case when the statistical weights (2.4) are parametrized by rational functions of the spectral parameters. The statistical weights are then given by

$$\begin{aligned} a^{XXX}(l_\alpha, \nu_k) &= l_\alpha - \nu_k + \tfrac{1}{2}\mathrm{i}c \\ b^{XXX}(l_\alpha, \nu_k) &= l_\alpha - \nu_k - \tfrac{1}{2}\mathrm{i}c \\ c^{XXX}(l_\alpha, \nu_k) &= -\mathrm{i}c \end{aligned} \tag{6.1}$$

where c is an arbitrary constant. In this case, the L-operator is constructed straightforwardly as above, see (3.1)

$$\mathsf{L}^{XXX}_{\alpha k}(l_\alpha - \nu_k) = l_\alpha - \nu_k - \tfrac{1}{2}\mathrm{i}c\sigma^\alpha_z\sigma^k_z - \mathrm{i}c(\sigma^\alpha_-\sigma^k_+ + \sigma^\alpha_+\sigma^k_-). \tag{6.2}$$

The following expression is the solution for the partition function with domain wall boundary conditions in the rational model

$$Z^{XXX}_N(\{l_\alpha\},\{\nu_j\}) = (-1)^N \frac{\prod_{j=1}^N \prod_{\alpha=1}^N (\nu_j - l_\alpha - \frac{1}{2}\mathrm{i}c)(\nu_j - l_\alpha + \frac{1}{2}\mathrm{i}c)}{\prod_{1\leqslant k<j\leqslant N}(\nu_k - \nu_j)\prod_{1\leqslant\beta<\alpha\leqslant N}(l_\alpha - l_\beta)} \det \mathcal{M}^{XXX} \tag{6.3}$$

where

$$\mathcal{M}^{XXX} = \frac{ic}{(\nu_j - l_\alpha - \frac{1}{2}\mathrm{i}c)(\nu_j - l_\alpha + \frac{1}{2}\mathrm{i}c)}. \tag{6.4}$$

This formula can be proven using the same methods as above with lemmas (a), (c) and (d) replaced by the following lemmas.

Lemma. (a^{XXX}) $Z^{XXX}_1 = -\mathrm{i}c$

Lemma. (c^{XXX}) Z^{XXX}_N is a polynomial of degree $N-1$ in each variable l_α and ν_j separately.

Lemma. (d^{XXX}) The partition function obeys the following recursion formula:

$$\begin{aligned} Z^{XXX}_N(\{l_\alpha\},\{\nu_k\})\big|_{l_\beta-\nu_j=-\mathrm{i}c/2} &= -\mathrm{i}c\prod_{k\neq j}^N(\nu_k - l_\beta - \tfrac{1}{2}\mathrm{i}c) \\ &\quad\times \prod_{\alpha\neq\beta}^N(\nu_j - l_\alpha - \tfrac{1}{2}\mathrm{i}c) Z^{XXX}_{N-1}(\{l_{\alpha\neq\beta}\},\{\nu_{k\neq j}\}). \end{aligned} \tag{6.5}$$

Lemma (b) remains unaltered.

It should also be noted that (6.3) can be obtained from (5.1) by making the following substitutions

$$2\eta \to \epsilon c \qquad l \to \epsilon l \qquad \text{as} \quad \epsilon \to 0. \tag{6.6}$$

This then leads to an extra factor of ϵ^{N^2} in the partition function. This is easily understood since the above substitutions would change each statistical weight in (2.4) by a factor of ϵ.

7. Homogeneous lattice

The above results concentrated on the inhomogeneous lattice whereby with each vertex corresponded a different set of statistical weights, see (3.1) and (3.4), parametrized by $\{l_\alpha\}$ and $\{\nu_k\}$. In this section, we shall consider the special case of the homogeneous lattice where the statistical weight for each vertex is parametrized by the same number. More explicitly, we shall consider

$$l_\alpha \to l \qquad \nu_k \to \nu \qquad (\alpha, k = 1, \ldots, N).$$

Each vertex will then be parametrized by the difference $x = l - \nu$.

To derive the partition function for the homogeneous lattice, we will start with the partition function (5.1) for the inhomogeneous lattice. To make the steps of the proof more explicit, the partition function (5.1) is rewritten as

$$Z_N = (-\mathrm{i}\sin 2\eta)^N \frac{\prod_{\alpha=1}^{N}\prod_{k=1}^{N} \sinh(l_\alpha - \nu_k - \mathrm{i}\eta)\sinh(l_\alpha - \nu_k + \mathrm{i}\eta)}{\prod_{1\leqslant\alpha<\beta\leqslant N} \sinh(l_\beta - l_\alpha) \prod_{1\leqslant k<l\leqslant N} \sinh(\nu_k - \nu_l)}$$
$$\times \det \begin{vmatrix} \phi(l_1-\nu_1) & \ldots & \phi(l_1-\nu_k) & \ldots & \phi(l_1-\nu_N) \\ \vdots & & \vdots & & \vdots \\ \phi(l_\beta-\nu_1) & \ldots & \phi(l_\beta-\nu_k) & \ldots & \phi(l_\beta-\nu_N) \\ \vdots & & \vdots & & \vdots \\ \phi(l_N-\nu_1) & \ldots & \phi(l_N-\nu_k) & \ldots & \phi(l_N-\nu_N) \end{vmatrix} \tag{7.1}$$

where

$$\phi(l-\nu) = [\sinh(l-\nu+\mathrm{i}\eta)\sinh(l-\nu-\mathrm{i}\eta)]^{-1}. \tag{7.2}$$

In this form, it is evident that elements in the same row depend on the same value of l_β and elements in the same column depend on the same value of ν_l.

First we will only be concerned with evaluating the limit

$$l_\alpha \to l \qquad (\alpha = 1, \ldots, N) \tag{7.3}$$

and leave the set $\{\nu_k\}$ arbitrary but each ν_k unique. The functions $\phi(l_\alpha - \nu_k)$ can then be Taylor expanded about l when $l_\alpha \to l$. The formula $l_\beta = l + (l_\beta - l)$ will prove useful where $l_\beta - l$ is of infinitesimal order in the limit (7.3). It should be noted that in the limit (7.3), singularities will arise in the denominator of (7.1) as will be explained later.

For convenience, we set $l_1 = l$. First take the limit $l_2 \to l$ and use $l_2 = l + (l_2 - l)$ to perform a Taylor series expansion. Thus elements in the second row of the matrix in (7.1) are of the form

$$\phi(l_2 - \nu_j) = \phi(l - \nu_j) + (l_2 - l)\phi'(l - \nu_j) + \cdots$$

where the prime denotes differentiation with respect to the argument of ϕ. The first term can then be removed from the matrix by subtraction of the first row. Elementary row operations such as this do not change the value of the determinant. The second row now has elements of the form

$$(l_2 - l)\phi'(l - \nu_j) + \tfrac{1}{2}(l_2 - l)^2\phi''(l - \nu_j) + \cdots$$

where $l_2 - l$ can now be factored out of the determinant. In the denominator of (7.1), the function $\sinh(l_2 - l)$ induces a singularity in the limit $l_2 \to l$. However, this singularity is cancelled by the zero from the term $(l_2 - l)$ factored out above. The next order term disappears in the limit $l_2 \to l$. Therefore the first two rows of the matrix in (7.1) now read

$$\begin{vmatrix} \phi(l-\nu_1) & \cdots & \phi(l-\nu_k) & \cdots & \phi(l-\nu_N) \\ \phi'(l-\nu_1) & \cdots & \phi'(l-\nu_k) & \cdots & \phi'(l-\nu_N) \end{vmatrix}. \tag{7.4}$$

Next, the limit $l_3 \to l$ is taken. Thus the elements in the third row of (7.1) are expanded as

$$\phi(l_3 - \nu_j) = \phi(l - \nu_j) + (l_3 - l)\phi'(l - \nu_j) + \frac{(l_3 - l)^2}{2!}\phi''(l - \nu_j) + \cdots. \tag{7.5}$$

The first term in this expansion is removed by subtracting the first row from the third row. The second term is removed by multiplying the second row by $(l_3 - l)$ and subtracting. Thus, the third row is left with terms of the form

$$\frac{(l_3 - l)^2}{2!}\phi''(l - \nu_j) + \cdots.$$

Factoring out $(l_3 - l)^2/2!$, the zero in the denominator from $\sinh^2(l_3 - l)$ is cancelled in the limit $l_3 \to l$ leaving a factor of $1/2!$. Thus, (7.1) takes the intermediate form

$$Z_N = \frac{(-\mathrm{i}\sin 2\eta)^N}{2!} \frac{\prod_{\alpha=1}^{N}\prod_{k=1}^{N} \sinh(l_\alpha - \nu_k - \mathrm{i}\eta)\sinh(l_\alpha - \nu_k + \mathrm{i}\eta)}{\prod_{4\leqslant\alpha<\beta\leqslant N} \sinh(l_\beta - l_\alpha) \prod_{1\leqslant k<l\leqslant N} \sinh(\nu_k - \nu_l)}$$
$$\times \det \begin{vmatrix} \phi(l-\nu_1) & \cdots & \phi(l-\nu_k) & \cdots & \phi(l-\nu_N) \\ \phi'(l-\nu_1) & \cdots & \phi'(l-\nu_k) & \cdots & \phi'(l-\nu_N) \\ \phi''(l-\nu_1) & \cdots & \phi''(l-\nu_k) & \cdots & \phi''(l-\nu_N) \\ \phi(l_4-\nu_1) & \cdots & \phi(l_4-\nu_k) & \cdots & \phi(l_4-\nu_N) \\ \vdots & & \vdots & & \vdots \\ \phi(l_N-\nu_1) & \cdots & \phi(l_N-\nu_k) & \cdots & \phi(l_N-\nu_N) \end{vmatrix}$$
$$(l_1 = l_2 = l_3 = l). \tag{7.6}$$

Using the procedure shown for l_2 and l_3, the arbitrary row β can be examined when the previous $\beta - 1$ ls have been analysed in the limit (7.3). Elements in the βth row are expanded in the form

$$\phi(l_\beta - \nu_j) = \phi(l - \nu_j) + (l_\beta - l)\phi'(l - \nu_j) + \cdots + \frac{(l_\beta - l)^{\beta-1}}{(\beta - 1)!}\phi^{(\beta-1)}(l - \nu_j) + \cdots \tag{7.7}$$

when $l_\beta \to l$. The first term is removed by subtraction with the first row of (7.6). The second term is removed by multiplying the second row by $(l_\beta - l)$ and subtracting. All the terms up to $\mathcal{O}((l_\beta - l)^{\beta-2})$ are removed in a similar fashion using elementary row operations. Thus the only terms left in the βth row are of the form

$$\frac{(l_\beta - l)^{\beta-1}}{(\beta-1)!}\phi^{(\beta-1)}(l-\nu_j) + \mathcal{O}((l_\beta - l)^\beta). \tag{7.8}$$

Again the term $(l_\beta - l)^{\beta-1}/(\beta-1)!$ is factored out. A term of the form $\sinh^{\beta-1}(l_\beta - l)$ appears in the denominator. This term is singular and of the same order as the term factored out. In the limit $l_\beta \to l$, a factor of $1/(\beta-1)!$ remains. The additional term in (7.8) of $\mathcal{O}((l_\beta - l)^\beta)$ is still $\mathcal{O}(l_\beta - l)$ and goes to zero giving no contribution in the limit $l_\beta \to l$.

This procedure is then carried out for each l_α whereby the αth row aquires $\alpha - 1$ derivatives and a factor of $1/(\alpha-1)!$. The following form for the partition function is thus obtained in the limit (7.3), all ls are the same, but the set $\{\nu_k\}$ remains as above

$$Z_N = (-\mathrm{i}\sin 2\eta)^N \frac{\prod_{k=1}^{N} \sinh^N(l-\nu_k - \mathrm{i}\eta)\sinh^N(l-\nu_k+\mathrm{i}\eta)}{\left[\prod_{\alpha=1}^{N-1}\alpha!\right]\prod_{1\leqslant k<l\leqslant N}\sinh(\nu_k - \nu_l)}$$
$$\times \det\begin{vmatrix} \phi(l-\nu_1) & \cdots & \phi(l-\nu_k) & \cdots & \phi(l-\nu_N) \\ \phi'(l-\nu_1) & \cdots & \phi'(l-\nu_k) & \cdots & \phi'(l-\nu_N) \\ \vdots & & \vdots & & \vdots \\ \phi^{(\beta-1)}(l-\nu_1) & \cdots & \phi^{(\beta-1)}(l-\nu_k) & \cdots & \phi^{(\beta-1)}(l-\nu_N) \\ \vdots & & \vdots & & \vdots \\ \phi^{(N-1)}(l-\nu_1) & \cdots & \phi^{(N-1)}(l-\nu_k) & \cdots & \phi^{(N-1)}(l-\nu_N) \end{vmatrix}. \tag{7.9}$$

This formula represents the partition function when the horizontal space is homogeneous but the vertical space is not.

Next the limit

$$\nu_k \to \nu \qquad (k = 1, \ldots, N) \tag{7.10}$$

is considered. The formula $\nu_k = \nu - (\nu - \nu_k)$ is used to Taylor expand an arbitrary element

$$\phi^{(\beta)}(l-\nu_k) = \phi^{(\beta)}(l-\nu) + (\nu-\nu_k)\phi^{(\beta+1)}(l-\nu) + \cdots + \frac{(\nu-\nu_k)^{k-1}}{(k-1)!}\phi^{(\beta+k-1)}(l-\nu) + \cdots. \tag{7.11}$$

As above, the limits (7.10) are evaluated in order of increasing k with elementary column operations used instead. The zeros in the denominator of the form $\sinh^{k-1}(\nu - \nu_k)$ are removed by the factors $(\nu - \nu_k)^{k-1}/(k-1)!$ leaving an overall factor of $1/(k-1)!$ for the kth column. Thus the first column aquires no extra derivatives, the second column aquires one extra derivative and the kth column aquires $k-1$ additional derivatives. In addition, the factor $\prod_{k=1}^{N} \sinh^N(l - \nu_k - \mathrm{i}\eta)\sinh^N(l - \nu_k + \mathrm{i}\eta)$ from (7.9) becomes $\phi^{-N^2}(x)$ where $x = l - \nu$. Combining these results, the partition function in the homogeneous limit is given by

$$Z_N = \frac{(-\mathrm{i}\sin 2\eta)^N}{\phi^{N^2}(x)\left[\prod_{\alpha=1}^{N-1}\alpha!\right]^2} \times \det \begin{vmatrix} \phi(x) & \cdots & \phi^{(k-1)}(x) & \cdots & \phi^{(N-1)}(x) \\ \phi'(l-\nu_1) & \cdots & \phi^{(k)}(x) & \cdots & \phi^{(N)}(x) \\ \vdots & & \vdots & & \vdots \\ \phi^{(\beta-1)}(x) & \cdots & \phi^{(\beta+k-2)}(x) & \cdots & \phi^{(\beta+N-2)}(x) \\ \vdots & & \vdots & & \vdots \\ \phi^{(N-1)}(x) & \cdots & \phi^{(N+k-2)}(x) & \cdots & \phi^{(2N-2)}(x) \end{vmatrix} . \tag{7.12}$$

This can then be written in a more convenient way

$$Z_N = \frac{(-\mathrm{i}\sin 2\eta)^N \det\mathcal{H}}{\phi^{N^2}(x)\left[\prod_{k=1}^{N-1} k!\right]^2} \tag{7.13}$$

where the matrix $\mathcal{H}$ is given by

$$\mathcal{H}_{\alpha k} = \frac{\mathrm{d}^{\alpha+k-2}}{\mathrm{d}x^{\alpha+k-2}}\phi(x). \tag{7.14}$$

where α and k are matrix indices. This is the result for the homogeneous lattice. It should be noted that this procedure can be carried out for various other limits of the spectral parameters. For example, all the νs could be the same with a subset of the ls going to one value while the rest of the ls go to another value. Using the above method, many different situations can be analysed.

The partition function for the rational six-vertex model also has a homogeneous limit that is similar to (7.13). The following result is proven using the methods above

$$Z_N^{XXX} = \frac{(-\mathrm{i}c)^N \det\mathcal{H}^{XXX}}{\theta^{N^2}(x)\left[\prod_{k=1}^{N-1} k!\right]^2} \tag{7.15}$$

where

$$\mathcal{H}_{\alpha k}^{XXX} = \frac{\mathrm{d}^{\alpha+k-2}}{\mathrm{d}x^{\alpha+k-2}}\theta(x) \tag{7.16}$$

with the function $\theta(x)$ defined by

$$\theta(x) = \left[x^2 + \frac{c^2}{4}\right]^{-1} . \tag{7.17}$$

8. Conclusion

In this paper we showed that the partition function for the six-vertex model on a finite square lattice can be explicitly evaluated in the case of domain wall boundary conditions. The partition function was proved to be equal to the determinant of some matrix. The size of this matrix is equal to the size of the lattice. In general, the partition function of the six-vertex model was expressed in terms of trigonometric functions. The determinant representation changed essentially for the homogeneous case. In the case of the rational six-vertex model, the matrix elements are rational functions of the spectral parameters. Evaluation of the thermodynamic limit of the inhomogeneous model is not straightforward and remains an open problem. However, we expect the free energy of the six-vertex model with domain wall boundary conditions to differ from that with periodic boundary conditions since the model is critical and surface effects should influence macroscopic quantities.

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Appendix. Basics of QISM

In this appendix, we briefly review the QISM formulation of the XXZ Heisenberg antiferromagnet. The six-vertex model has the same algebraic stucture as the inhomogeneous Heisenberg anti-ferromagnet, XXZ model. Therefore we will review the important properties of QISM [14] in the context of the XXZ model.

The XXZ Hamiltonian is given by

$$H = -\sum_{n=1}^{M} \left[\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \Delta(\sigma_n^z \sigma_{n+1}^z - 1) + 2h\sigma_n^z\right] \tag{A1}$$

where h is the external magnetic field and σ_n^j are standard Pauli matrices at the nth site of the lattice. There are a total of M sites. Δ is the anisotropy parameter. $\Delta = 1$ for the XXX model and $\Delta = \cos 2\eta$ for the XXZ model. We impose periodic boundary conditions such that $\sigma_1 = \sigma_{M+1}$. This system has been solved by means of both the coordinate and algebraic Bethe ansatz. The algebraic solution comes from the QISM formulation of the model.

The transition matrix $\mathbf{T}(n, m|l)$ from site m to site $n+1$ is the product of $\mathbf{L}$-operators

$$\mathbf{T}(n, m|l) = \mathbf{L}_n(l)\mathbf{L}_{n-1}(l)\cdots\mathbf{L}_j(l)\cdots\mathbf{L}_{m+1}(l)\mathbf{L}_m(l) \tag{A2}$$

where l is the spectral parameter, rapidity. These $\mathbf{L}$-operators obey the following intertwining relation with the $\mathbf{R}$-matrix (A12)

$$\mathbf{R}(l,\mu)\left[\mathbf{L}_n(l)\otimes\mathbf{L}_n(\mu)\right] = \left[\mathbf{L}_n(\mu)\otimes\mathbf{L}_n(l)\right]\mathbf{R}(l,\mu). \tag{A3}$$

Thus we see that the **R**-matrix acts in the tensor product space of the two **L**-operators. This can be made more explicit by rewriting (A3) as

$$\widetilde{\mathbf{R}}_{\alpha\beta}(l,\mu)\mathbf{L}_{\alpha n}(l)\mathbf{L}_{\beta n}(\mu) = \mathbf{L}_{\beta n}(\mu)\mathbf{L}_{\alpha n}(l)\widetilde{\mathbf{R}}_{\alpha\beta}(l,\mu) \tag{A4}$$

where subindices enumerate the spaces where the matrices act non-trivially. The matrix $\widetilde{\mathbf{R}}$ is defined by $\widetilde{\mathbf{R}} = \mathbf{PR}$ where $\mathbf{P}$ is the permutation matrix which interchanges two spaces and is given explicitly by

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{A5}$$

Here $\mathbf{L}_{\alpha n}$ is a 2×2 matrix which acts in the two-dimensional space α and $\mathbf{L}_{\beta n}$ acts in the space β, n is still the lattice site. The transition matrix also obeys a similar relationship with the **R**-matrix.

The transition matrix through the entire lattice is called the monodromy matrix $\mathbf{T}(l)$

$$\mathbf{T}(l) \equiv \mathbf{T}(M,1|l) = \mathbf{L}_M(l)\cdots\mathbf{L}_1(l) = \begin{pmatrix} A(l) & B(l) \\ C(l) & D(l) \end{pmatrix} \tag{A6}$$

where A,B, C and D are operators which act on a pseudovacuum, $|0\rangle$, in the following way:

$$\begin{aligned} A(l)|0\rangle &= a(l)|0\rangle \qquad & C(l)|0\rangle &= 0 \\ D(l)|0\rangle &= d(l)|0\rangle \qquad & \langle 0|B(l) &= 0. \end{aligned} \tag{A7}$$

The pseudovacuum for the above Hamiltonian (A1), $|0\rangle$, is the state with all spins up, $|\uparrow\rangle$. The vacuum eigenvalues $a(l)$ and $d(l)$ are c-valued functions of the spectral parameter. Their explicit form depends on the model in question. The XXX model has vacuum eigenvalues

$$a(l) = \left(l - \tfrac{1}{2}\mathrm{i}c\right)^M \qquad d(l) = \left(l + \tfrac{1}{2}\mathrm{i}c\right)^M \tag{A8}$$

and the XXZ model has vacuum eigenvalues

$$a(l) = \sinh^M(l - \mathrm{i}\eta) \qquad d(l) = \sinh^M(l + \mathrm{i}\eta). \tag{A9}$$

Here c and η are coupling constants. $B(l)$ and $C(l)$ can be considered as creation and annihilation operators on the vacuum. Hence, the eigenfunction of the Hamiltonian (A1) for N spins down ($N < M$) is of the form

$$|\Psi_N\{l_j\}\rangle = \prod_{j=1}^{N} B(l_j)|0\rangle. \tag{A10}$$

The monodromy matrix (A5) obeys the following intertwining relation

$$\widetilde{\mathbf{R}}_{\alpha\beta}(l,\mu)\mathbf{T}_\alpha(l)\mathbf{T}_\beta(\mu) = \mathbf{T}_\beta(\mu)\mathbf{T}_\alpha(l)\widetilde{\mathbf{R}}_{\alpha\beta}(l,\mu) \tag{A11}$$

where $\mathbf{T}_\alpha(l)$ acts on the two-dimensional space α and $\mathbf{T}_\beta(\mu)$ acts on the space β. From (A10) it follows that ($[B(l), B(\mu)] = 0$ and $[C(l), C(\mu)] = 0$). The trace of $\mathbf{T}(l)$ is the transfer matrix $\tau(l)$ which is the generator of all conservation laws. From (A10) we see that the transfer matrices commute for different values of the spectral parameter, $[\tau(l), \tau(\mu)] = 0$.

The $\mathbf{R}$-matrix for the above Hamiltonian is of the following form

$$\mathbf{R}(l,\mu) = \begin{pmatrix} f(\mu,l) & 0 & 0 & 0 \\ 0 & g(\mu,l) & 1 & 0 \\ 0 & 1 & g(\mu,l) & 0 \\ 0 & 0 & 0 & f(\mu,l) \end{pmatrix} \tag{A12}$$

where for the XXX model

$$f(\mu,l) = 1 + \frac{\mathrm{i}c}{\mu - l} \qquad g(\mu,l) = \frac{\mathrm{i}c}{\mu - l} \tag{A13}$$

and for the XXZ model

$$f(\mu,l) = \frac{\sinh(\mu - l + 2\mathrm{i}\eta)}{\sinh(\mu - l)} \qquad g(\mu,l) = \frac{\mathrm{i}\sin(2\eta)}{\sinh(\mu - l)} \tag{A14}$$

where c and η are coupling constants, $c > 0$ and $\Delta = \cos 2\eta$.

The above intertwining relation for the $\mathbf{L}$-operator has a solution whose result for the XXX model is

$$\mathbf{L}_n(l) = l - \frac{\mathrm{i}c}{2}\begin{pmatrix} \sigma_z^n & 2\sigma_-^n \\ 2\sigma_+^n & -\sigma_z^n \end{pmatrix} \tag{A15}$$

while the result for the XXZ model is

$$\mathbf{L}_n(l) = \begin{pmatrix} \sinh(l - \mathrm{i}\eta\sigma_z^n) & -\sigma_-^n \sin(2\eta) \\ -\sigma_+^n \sin(2\eta) & \sinh(l + \mathrm{i}\eta\sigma_z^n) \end{pmatrix}. \tag{A16}$$

These solutions can be generalized to the inhomogeneous case whereby the spectral parameter is shifted by a number ν_n which depends on the site of the lattice. The monodromy matrix then takes the following form

$$\mathbf{T}(l) = \mathbf{L}_M(l - \nu_M)\cdots\mathbf{L}_1(l - \nu_1) \tag{A17}$$

and still satisfies the intertwining relation (A10) above. In this case, the vacuum eigenvalues (A7) for the inhomogeneous XXX model become

$$a(l) = \prod_{j=1}^{M}(l - \nu_j - \tfrac{1}{2}\mathrm{i}c) \qquad d(l) = \prod_{j=1}^{M}(l - \nu_j + \tfrac{1}{2}\mathrm{i}c) \tag{A18}$$

and the vacuum eigenvalues (A8) for the inhomogeneous XXZ model become

$$\begin{aligned} a(l) &= \prod_{j=1}^{M}\sinh(l - \nu_j - \mathrm{i}\eta) \\ d(l) &= \prod_{j=1}^{M}\sinh(l - \nu_j + \mathrm{i}\eta). \end{aligned} \tag{A19}$$

The inhomogeneous form of (A15) was used in the construction of the partition function of the six-vertex model.

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